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Energy distribution from vertical impact of a three-dimensional solid body onto the flat free surface of an ideal fluid

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Abstract

Hydrodynamic impact phenomena are three dimensional in nature and naval architects need more advanced tools than a simple strip theory to calculate impact loads at the preliminary design stage. Three-dimensional analytical solutions have been obtained with the help of the so-called inverse Wagner problem as discussed by Scolan and Korobkin in 2001. The approach by Wagner provides a consistent way to evaluate the flow caused by a blunt body entering liquid through its free surface. However, this approach does not account for the spray jets and gives no idea regarding the energy evacuated from the main flow by the jets. Clear insight into the jet formation is required. Wagner provided certain elements of the answer for two-dimensional configurations. On the basis of those results, the energy distribution pattern is analysed for three-dimensional configurations in the present paper.

1. Introduction

The three-dimensional hydrodynamic impact problem is considered within the classical assumptions of the Wagner theory. A blunt solid body initially touches a flat free surface at a single point which is taken as the origin of the Cartesian coordinate system Oxyz [see Fig. 1(a)]. Then the body starts to enter the liquid vertically at a prescribed velocity $U(t)$ [see Fig. 1(b)]. The liquid is assumed to be ideal and incompressible and the corresponding flow to be potential. External mass forces and capillary effects are not taken into account. The impact phenomenon is very rapid; during a drop test, the maximum of the body deceleration is usually much higher than the acceleration due to gravity.

During the initial stage of impact, the flow region is divided into three parts: (i) outer region, (ii) jet root region and (iii) jet region. These subdomains of the flow region are shown in Fig. 1(c) for a two-dimensional body. For blunt bodies, the curvature of the liquid free surface in the jet root region is very large but the dimension of this region during the initial stage is much smaller than the characteristic dimension of the outer region (Cointe, 1989). Therefore, in order to investigate the liquid flow in the outer region, the moving jet root region can be approximately replaced by a corresponding curve $\Gamma_b(t)$ on the entering body surface [see Fig. 1(d)]. This curve now defines an approximate wet surface of the body denoted $D_b(t)$. This replacement of the narrow jet root region by the curve $\Gamma_b(t)$ cuts the spray jet from the outer region, where the flow can be determined independent of what might occur in both the jet root region and the jet region itself.

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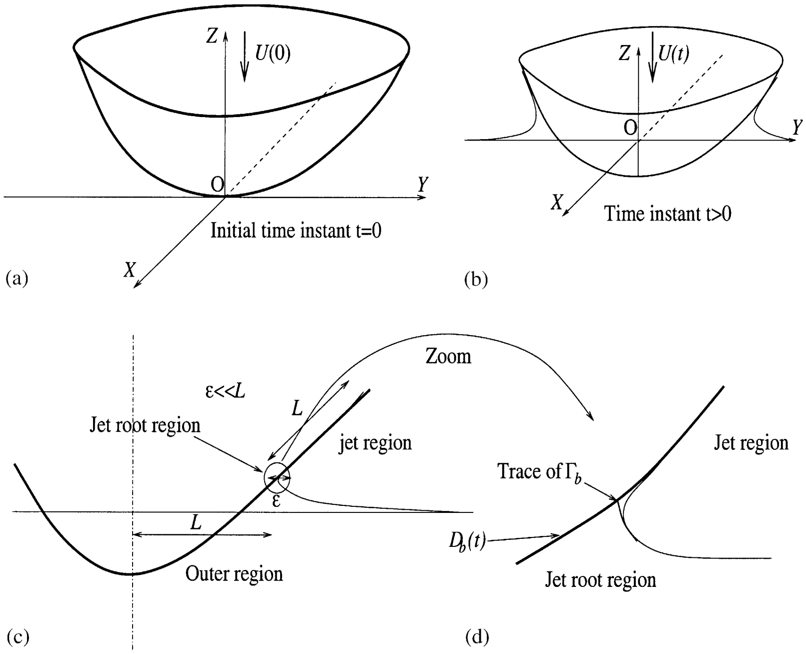

Fig. 1. (a) Coordinate system in which the body shape is described; the horizontal plane corresponds to the initially flat free surface. (b) Local deformations of free surface as the body enters the fluid. The relative dimensions of the deformations are not correct and intentionally stretched for sake of illustration. (c) The three regions around the jet root with their characteristic lengths. The size of the jet root region ε is very small compared to the length scales of the other regions and hence curvature is locally very large in the jet root. (d) Approximate curve $\Gamma_b(t)$ which cuts the spray and encloses the approximate wet surface $D_b(t)$.

Further simplifications for the outer region suggested by Wagner (1932) are available at the initial stage of blunt body impact. First, the elevation of the three-dimensional curve $\Gamma_b(t)$ above the plane $z = 0$ is of the order of the penetration depth which is much smaller than the horizontal dimensions of the part of the body surface $D_b(t)$. Therefore, the curve $\Gamma_b(t)$ can be approximately replaced during the initial stage by its projection $\Gamma(t)$ on the plane $z = 0$ (see Fig. 2). The curve $\Gamma(t)$ is known within the Wagner approach as the contact line. Second, the wet part of the body $D_b(t)$ can be approximated with the flat disc $D(t)$ bounded by the contact line $\Gamma(t)$. The flat disc $D(t)$ is known within the Wagner approach as the contact region between the entering body and the liquid. The Wagner approach itself is sometimes referred to as the "flat-disc approximation". Third, in order to describe the liquid flow in the outer region, the boundary conditions on both the free surface of the liquid and on the wet part of the body can be linearized and imposed on the undisturbed initial position of the liquid boundary $z = 0$. This means that the free surface and the wet part of the body must be projected one-to-one onto the horizontal plane $z = 0$. This is illustrated in the lower sketch of Fig. 2. The original nonlinear problem is reduced within the Wagner theory to a linear mixed boundary-value problem for the velocity potential $\phi(x, y, z, t)$. The body surface is impermeable and the Neumann condition is prescribed on the wet body surface, $\phi_z = -U(t)$. Hence, the dynamic free surface condition reduces to the homogeneous Dirichlet condition, $\phi = 0$. As a consequence, the flow region can be extended symmetrically with respect to the plane $z = 0$ containing both the linearized free surface and wet body surface. One finally has to calculate the flow over a corresponding flat disc and to find the shape of the disturbed free surface using the kinematic condition.

Within the direct Wagner problem (both the shape and the velocity of the body are given), the position of $\Gamma(t)$ must be calculated using the Wagner condition, which is not an easy task. The Wagner condition requires the continuity between the disturbed free surface and the surface of the entering body along the contact line. Alternatively within the inverse Wagner problem [the time variation of the contact line $\Gamma(t)$ is known and the body velocity as well], the body shape which provides this contact line can be reconstructed with the help of the Wagner condition.

Since the pioneering works by Wagner (1932) and then by Borodich (1988), the inverse problem has received little attention. However, among the major advances in that domain, one may cite exact solutions of the direct Wagner problem after calculating analytical solutions from the inverse Wagner problem. For example, the analytic solution of

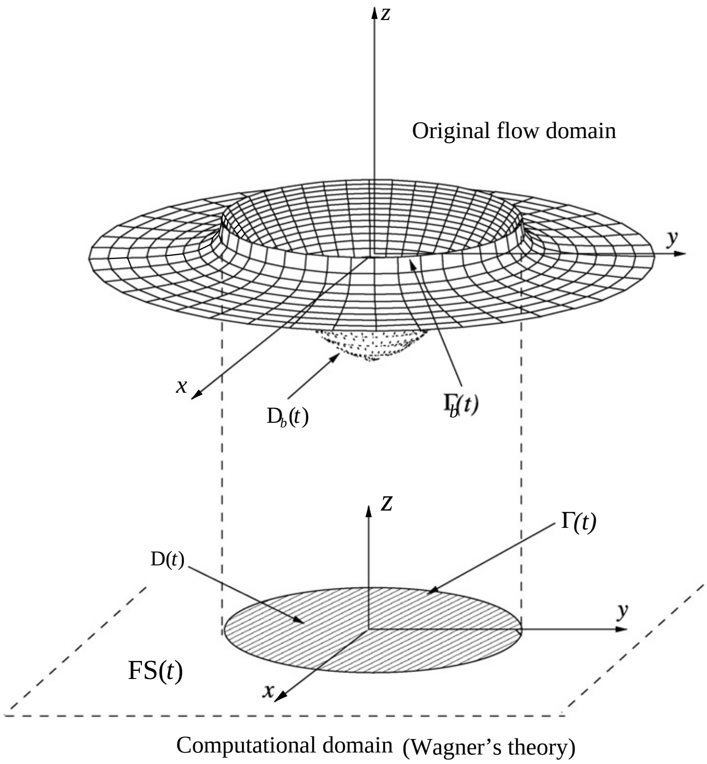

Fig. 2. Sketches of three-dimensional flow pattern for normal penetration of a blunt body into a liquid within the original problem and within the linearized Wagner approximation: $D(t)$, contact region; $FS(t)$, liquid free surface; $\Gamma(t)$, contact line.

the elliptic paraboloid entering a liquid with a constant speed is now a reference case for the validation of other numerical approaches. Furthermore, many families of shapes for which analytical solutions can be calculated have been designed (Scolan and Korobkin, 2000).

More generally, algorithms for the generation of shapes with prescribed constraints can now be established and used in a standard way provided that the contact line $\Gamma(t)$ is elliptic. The case of elliptic contact lines might be considered as too restrictive since an ellipse depends on only two parameters: the two semi-axes. In spite of that, since the time variation of $\Gamma(t)$ may be arbitrary, many shapes of practical interest can be studied as long as the body is blunt around its initial contact point, which is the main assumption of the Wagner theory. In many respects, this is the case of the bow flare of tankers and various parts of sailing boats (bow, stern and sides).

The Wagner theory is in fact formally valid during the initial stage, when the penetration depth of the entering body is much smaller than the dimensions of its wet part. But close to the contact line, the theory fails since both the liquid velocity and the hydrodynamic pressure have singularities along this line. The singularity is integrable and hence the force can be calculated. However, in order to get uniformly valid pressure distribution and to improve prediction of the hydrodynamic force on the entering body, a solution which describes details of the flow close to the contact line must be introduced.

The jet root region was analysed by Wagner (1932) and a two-dimensional potential solution exists. So far the problem posed in the flow region requires CFD approaches and actually the physical phenomena are reasonably reproduced by using smooth particle hydrodynamics or volume of fluid algorithms that are well suited to simulations of rapid phenomena [see recent results by Fontaine et al. (2000)].

Using much simpler approaches, Wagner derived an analytical two-dimensional solution for the jet root region within the potential theory. An infinite length of the jet was theoretically predicted. For three-dimensional bodies one must even deal with a jet sheet. In order to obtain the shape of this jet sheet and the flow inside it, the jet region also has to be considered. The jet solution has then to be matched with that for the jet root region. In the two-dimensional wedge-entry problem, the jet solution was derived by Howison et al. (1991). It was shown that the flow in the jet region is governed by the shallow-water equations and the jet is wedge shaped. This technique was extended by Korobkin (1994, 1997) to the case of a parabolic contour entering a compressible liquid. Using both the known liquid flow in the jet region and the geometry of this region, the energy of the jet was evaluated in both plane and axisymmetric cases. It was shown that during the impact of two-dimensional or axisymmetric blunt bodies onto a compressible liquid free surface at a constant velocity, half of the work done to move the body goes to the main flow kinetic energy and the other half is taken away with spray jets. The jets are very thin at the initial stage but the jet velocity far exceeds the velocity of the entering body.

This result was confirmed by Molin et al. (1996) using another method for the two-dimensional problem of impact onto an incompressible liquid surface. This method is based on the concept of energy flux evaluated through the jet root region. The main advantage of this approach is that the flux can be directly determined from the solution in the jet root region and there is no need to deal with the flow in the jet region and its geometry. This approach is used in the present paper to evaluate the part of the energy taken away with the jet in the three-dimensional impact problem. It will be shown that in order to evaluate the jet energy, one needs only to know the asymptotic behaviour of the outer solution close to the contact line.

The outer solution for an arbitrary shape of three-dimensional entering body is still not available even within the Wagner theory. The present study is restricted to an elliptic contact line for which the velocity potential is known and the so-called inverse Wagner problem has solutions. It is important to note that there are no restrictions on the evolution of the semi-axes of the elliptic contact line in time.

It is shown that the outer flow is approximately two dimensional close to elliptic contact lines. Therefore, it is possible to use the two dimensional nonlinear solution by Wagner (1932) for the jet root region. By matching locally the three-dimensional outer solution with the two dimensional jet root solution, one arrives at a uniformly valid asymptotic description of the pressure distribution. In the case of an elliptic contact region, this combined solution is used to evaluate the energy distribution throughout the flow domain and to prove that the energy is equally transmitted to the bulk of the fluid and to the spray jet in the case of constant velocity of the entering body.

2. Asymptotic analysis close to the contact line

Within the Wagner theory, the wet part of the entering body is approximated by a flat disc $D(t)$ which evolves in time. The boundary conditions are linearized and imposed on the initially undisturbed liquid level $z = 0$. The liquid flow caused by the impact is assumed as irrotational and is described by the velocity potential $\phi_{\text{out}}(x, y, z, t)$, where $z < 0$. It is assumed that the Wagner problem has been solved already so that the region $D(t)$ and the body velocity $U(t)$ are prescribed. We restrict ourselves to the case of elliptic contact regions, $D(t) = \{x, y \mid x^2/a^2(t) + y^2/b^2(t) < 1\}$, with the planar ($a/b \to 0$) and axisymmetric ($a/b = 1$) problems representing the limiting cases. Here $a(t)$, $b(t)$ and $U(t)$ are arbitrary monotonic positive functions, which satisfy the following inequalities $a(t) \leq b(t)$, $U(t) \ll \dot{a}(t)$ and $b(0) = 0$ according to the basic assumptions of the Wagner theory. The aspect ratio is denoted $k(t) = a(t)/b(t)$ and the eccentricity is $e = \sqrt{1 - k^2}$. The dot stands for the time derivative.

The easiest way to present the velocity potential $\phi_{\text{out}}(x, y, z, t)$ in the case of elliptic contact region is to treat it as the limiting case of a vertically moving ellipsoid in an unbounded fluid. By using ellipsoidal coordinates as defined either in Milne-Thomson (1960, Art. 17.50–17.52) or in Lamb (1932, Art. 112–114), we obtain [see Scolan and Korobkin (2001) for further details]

$$\phi_{\text{out}}(x, y, z, t) = \frac{U(t)a^2(t)b(t)z}{2E(e)} \int_{\lambda(x,y,z,t)}^{\infty} \frac{\mathrm{d}\sigma}{\sigma^{3/2}(a^2 + \sigma)^{1/2}(b^2 + \sigma)^{1/2}} \quad (z < 0), \tag{1}$$

where $e(t) = \sqrt{1 - a^2(t)/b^2(t)}$, $E(e)$ is the complete elliptic integral of the second kind

$$E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, \mathrm{d}\theta, \tag{2}$$

and $\lambda(x, y, z, t)$ is the nonnegative root of the cubic equation

$$a^{-2}b^{-2}\lambda^3 + L_2\lambda^2 + L_1\lambda - z^2 = 0, \tag{3}$$

$$L_1(x, y, z, t) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{a^2 + b^2}{a^2b^2} z^2, \quad L_2(x, y, z, t) = \frac{1}{a^2} + \frac{1}{b^2} - \frac{x^2 + y^2 + z^2}{a^2b^2}. \tag{4}$$

The elliptic disc $D(t)$ corresponds to $\lambda(x, y, 0, t) = 0$. The integral in Eq. (1) is singular as $\lambda \to 0$. In order to study the behaviour of the outer velocity potential $\phi_{\text{out}}(x, y, z, t)$ close to the periphery of the contact region, an integration by part is performed

$$\phi_{out}(x,y,z,t) = \frac{Ua^2b}{E(e)\sqrt{(a^2+\lambda)(b^2+\lambda)}}\frac{z}{\sqrt{\lambda}} - \frac{Ua^2bz}{2E(e)}\int_\lambda^\infty \frac{(a^2+b^2+2\sigma)\,\mathrm{d}\sigma}{\sigma^{1/2}(a^2+\sigma)^{3/2}(b^2+\sigma)^{3/2}}. \tag{5}$$

Using the definition of the ellipsoidal coordinates, we obtain

$$\frac{z}{\sqrt{\lambda}} = -\sqrt{1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}} \tag{6}$$

in the flow region, $z<0$, which finally gives

$$\phi_{out} = -\frac{Ua^2b}{E(e)\sqrt{(a^2+\lambda)(b^2+\lambda)}}\sqrt{1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}} - \frac{Ua^2bz}{2E(e)}\int_\lambda^\infty \frac{(a^2+b^2+2\sigma)\,\mathrm{d}\sigma}{\sigma^{1/2}(a^2+\sigma)^{3/2}(b^2+\sigma)^{3/2}}. \tag{7}$$

Expression (7) is suitable to analyse the local behaviour of the outer solution near the contact line $\Gamma(t) = \{x,y \mid x = a(t)\cos\alpha,\ y = b(t)\sin\alpha,\ 0\leq\alpha<2\pi\}$, where α is a parameter.

For that purpose, it is convenient to introduce the local coordinate system (P,x_1,y_1,z_1), where $x = a\cos\alpha + x_1$, $y = b\sin\alpha + y_1$, $z = z_1$ and $(x_1^2+y_1^2+z_1^2)/a^2 = \mathcal{O}(\epsilon^2)$, with ϵ being a small nondimensional parameter such that $\epsilon \ll 1$ and which formally characterizes the size of the contact line vicinity. Within the local coordinate system the following asymptotic formulae for the coefficients in Eq. (4) are valid:

$$L_1 = -2s(x_1,y_1,\alpha,t)[1+\mathcal{O}(\epsilon)], \quad s(x_1,y_1,\alpha,t) = x_1a^{-1}\cos\alpha + y_1b^{-1}\sin\alpha, \tag{8}$$

$$L_2 = \mu(\alpha,t)[1+\mathcal{O}(\epsilon)], \quad \mu(\alpha,t) = a^2(t)\sin^2\alpha + b^2(t)\cos^2\alpha, \tag{9}$$

where $\mu/a^2 = \mathcal{O}(1)$ and $s = \mathcal{O}(\epsilon)$ in the leading order as $\epsilon\to 0$. Taking into account formulae (8) and (9), we obtain the positive root of Eq. (3) in the form

$$\lambda = \frac{a^2b^2}{\mu}\left(s+\sqrt{s^2+\frac{\mu z_1^2}{a^2b^2}}\right)[1+\mathcal{O}(\epsilon^2)] \quad (\epsilon\to 0). \tag{10}$$

The second term in formula (7) is of the order of $\mathcal{O}(\epsilon)$. In the first term, we have $\lambda/a^2 = \mathcal{O}(\epsilon)$, which follows from Eq. (10), and

$$1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda} = -2s+\frac{\lambda\mu}{a^2b^2}+\mathcal{O}(\epsilon^2). \tag{11}$$

By using asymptotic formulae (10) and (11), we finally obtain the local behaviour of the velocity potential close to the contact line as

$$\phi_{out}(x,y,z,t) = -\frac{Ua}{E(e)}\left(\sqrt{s^2+\frac{\mu z_1^2}{a^2b^2}}-s\right)^{1/2}[1+\mathcal{O}(\epsilon^{1/2})], \tag{12}$$

where $s = s(x_1,y_1,\alpha,t)$ and $z_1 = z$.

Let us consider a new coordinate system (P,ξ,η,ζ), which is obtained by rotation of the system (P,x_1,y_1,z_1) counterclockwise by the angle α so that $x_1 = \xi\cos\alpha - \eta\sin\alpha$, $y_1 = \xi\sin\alpha + \eta\cos\alpha$ and $z_1 = \zeta$. It is worth noting that $s = \xi\sqrt{\mu}/(ab)$ in the new coordinate system, which implies that the flow is locally two dimensional. One should also note that the leading order term of the outer velocity potential in Eq. (11) does not depend on the coordinate η but only on ξ and ζ. Thus near the contact line, where $\sqrt{\xi^2+\zeta^2}/a = \mathcal{O}(\epsilon)$, the velocity potential of the outer flow behaves as

$$\phi_{out} = -UE^{-1}(e)\mu^{1/4}(a/b)^{1/2}(\sqrt{\xi^2+\zeta^2}-\xi)^{1/2}[1+\mathcal{O}(\epsilon^{1/2})]. \tag{13}$$

It is clear that the axes $P\xi$ and $P\eta$ are in normal and tangential directions to the contact line, respectively (this is described in Fig. 3). Therefore, near the contact line the flow in the tangential direction to this line (that is to say the direction $P\eta$) is negligible compared to the flow in the normal direction. The local flow described by Eq. (13) is similar to that in the corresponding two dimensional problem (see Wagner, 1932) but now the coefficient $E^{-1}(e)\mu^{1/4}(a/b)^{1/2}$ is strongly dependent on the three-dimensional configuration of the original problem. This two dimensional local flow can be matched to the solution in the jet root region established by Wagner (1932) for the planar impact problem.

3. Parameters of the jet in three-dimensional impact problem

The two dimensional flow in the jet root region was analysed by Wagner (1932), Cointe and Armand (1987) and Howison et al. (1991) among others. In this section, we do not reproduce the developed theory but only the main formulae yielding the parameters of the jet.

The flow in the jet root region is considered within the moving coordinate system (P, ξ, η, ζ), where $\sqrt{\xi^2+\eta^2+\zeta^2}/a \ll 1$. The local velocity potential $\phi_{\mathrm{root}}(\xi, \zeta)$ does not depend on the tangential coordinate η, which follows from the matching condition of the outer solution [Eq. (13)] to the solution in the jet root region.

The flow in the jet root region is quasi-stationary in the leading order as the size of this region tends to zero. This region is characterized by the jet thickness $\delta(\alpha, t)$ and the velocity $V(\alpha, t)$ of the fluid in the jet as illustrated in Fig. 4. The dynamic boundary condition on the free surface shows that the jet velocity $V(\alpha, t)$ is equal to the normal velocity of the point P, which is the origin of the moving coordinate system [see Cointe (1989) for more details]. The two dimensional jet root solution by Wagner (1932) provides, in particular, the asymptotics of both the velocity potential and the pressure on the body surface $\zeta = 0$:

$$\phi_{\mathrm{root}}(\xi, 0) \sim -4V\sqrt{\frac{\delta|\xi|}{\pi}}, \quad p_{\mathrm{root}}(\xi, 0) \sim 2\rho V^2\sqrt{\frac{\delta}{\pi|\xi|}}, \tag{14}$$

in the far field, where $|\xi|/\delta \gg 1$, $|\xi|/a \ll 1$ and $\zeta = 0$. Expressions (14) have to be considered as the "outer" asymptotics of the "inner" solution and matched to the "inner" asymptotics (13) of the "outer" solution. Comparing asymptotic formulae (13) and (14), one obtains the jet thickness as

$$\delta(\alpha, t) = \frac{\pi}{8}\frac{U^2(t)(a/b)\mu^{1/2}(\alpha, t)}{E^2(e)V^2(\alpha, t)}. \tag{15}$$

Details of the matching procedure were given by Cointe (1989).

The jet velocity $V(\alpha, t)$ is equal to the normal velocity of the moving contact line, the position of which is described by the equation $G(x, y, t) = 0$, where $G(x, y, t) = 1 - x^2/a^2(t) - y^2/b^2(t)$, yielding

$$V(\alpha, t) = \frac{\dot{G}}{|\nabla G|}, \quad \dot{G}(\alpha, t) = 2\frac{\dot{a}}{a}\cos^2\alpha + 2\frac{\dot{b}}{b}\sin^2\alpha, \quad |\nabla G|(\alpha, t) = \frac{2\mu^{1/2}(\alpha, t)}{a(t)b(t)}, \tag{16}$$

where ∇ is the gradient operator. Eqs. (15) and (16) make it possible to calculate the jet velocity $V(\alpha, t)$ and the jet thickness $\delta(\alpha, t)$ at any point along the contact line $(0 \leq \alpha < 2\pi)$ during the initial stage of the impact. These equations

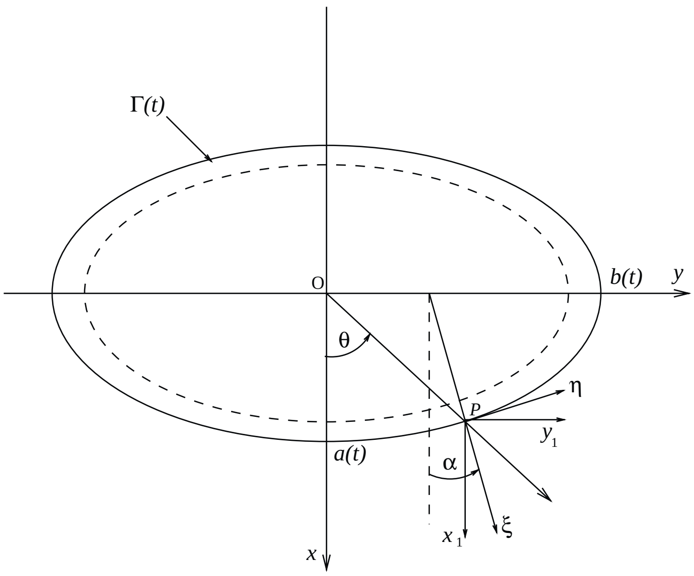

Fig. 3. Coordinate systems in the vicinity of the contact line $\Gamma(t)$.

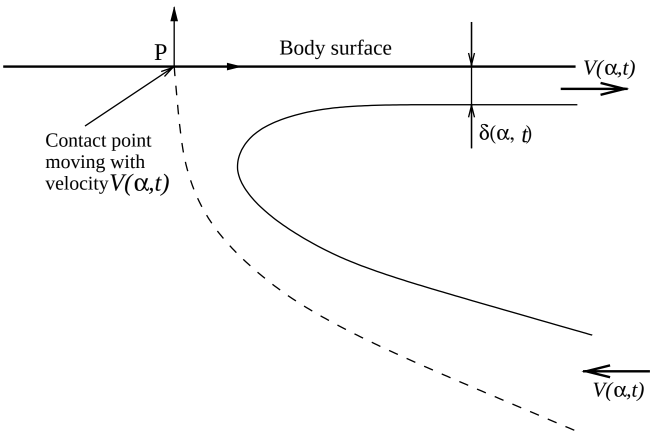

Fig. 4. Parameters which characterize the jet region: jet thickness $\delta(\alpha, t)$ and the velocity $V(\alpha, t)$ of the fluid in the jet.

lead to the helpful equality

$$\delta(\alpha, t)V^3(\alpha, t) = \frac{\pi}{16}\frac{U^2a^2}{E^2(e)}\dot{G}(\alpha, t) \tag{17}$$

used below to evaluate the flux of kinetic energy through the jet.

4. Flux of energy through the jet

The total velocity of fluid in the jet $V_f(\alpha, t)$ is equal to the jet velocity $V(\alpha, t)$ plus the normal velocity of the moving contact line, which yields $V_f(\alpha, t) = 2V(\alpha, t)$. The part of the kinetic energy $\Delta E_j(\alpha, t)$, which leaves the main flow region through the jet root region of a small length $\Delta\ell$ during a small time interval Δt, is given as

$$\Delta E_j(\alpha, t) = \frac{1}{2}\Delta m(\alpha, t)V_f^2(\alpha, t), \quad \Delta m(\alpha, t) = \rho\delta(\alpha, t)\Delta\ell V(\alpha, t)\Delta t, \tag{18}$$

where ρ is the liquid density and $\Delta\ell = \mu^{1/2}(\alpha, t)\Delta\alpha$. Eqs. (17) and (18) provide the total flux of the kinetic energy through the three-dimensional jet in the form

$$\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t} = 2\rho\int_0^{2\pi}\delta(\alpha, t)V^3(\alpha, t)\mu^{1/2}(\alpha, t)\,\mathrm{d}\alpha = \frac{\pi}{8}\frac{\rho U^2a^2}{E^2(e)}\int_0^{2\pi}\dot{G}(\alpha, t)\mu^{1/2}(\alpha, t)\,\mathrm{d}\alpha. \tag{19}$$

By using Eqs. (9) and (16), the integral in Eq. (19) can be evaluated analytically and the total flux is

$$\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t} = \frac{\rho U^2(t)\pi a}{E(e)}\left[\dot{a}b + \frac{1}{3}(\dot{b}a - \dot{a}b)\left(1 + k^2\frac{D(e)}{E(e)}\right)\right], \tag{20}$$

where $D(e)$ is the complete elliptic integral of the third kind.

5. Homothetical case

It is worth noting that in the homothetical case the elliptic disc $D(t)$ expands in such a way that the aspect ratio $k = a(t)/b(t)$ does not vary in time, Eq. (20) provides a much simpler expression for the flux

$$\left.\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t}\right|_{\dot{k}=0} = \frac{\rho U^2(t)\pi k^2b^2\dot{b}}{E(e)}. \tag{21}$$

In the homothetical case, where $\dot{a}/a = \dot{b}/b$, the derivative $\dot{G}(\alpha, t)$ given by Eq. (16) is equal to $2\dot{b}/b$ and the product $\delta(\alpha, t)V^3(\alpha, t)$ given by Eq. (17) does not depend on the angle α. Therefore, the quantity $\Delta E_j(\alpha, t)$ introduced by Eq. (18) does not depend on the angle α either. This implies that the amount of kinetic energy which leaves the main flow through any interval of the contact line does not depend on the position of this interval along the contact line but only on the interval length.

The two dimensional limiting case, $b(t) \gg a(t)$, can be treated as the homothetical one with $k = 0$. Eq. (21) is of little help now because it predicts an infinite flux as $k \to 0$, $b \to \infty$ and $kb(t) = a(t)$. This is the quantity $\Delta E_j(\alpha, t)/(\Delta \ell \Delta t)$ which is of main interest in the two dimensional impact problem. Eqs. (18) yield $\Delta E_j/(\Delta \ell \Delta t) = 2\rho \delta V^3$, where $\delta V^3 = \frac{1}{8}\pi U^2(t)a(t)\dot{a}(t)$ as it follows from Eqs. (16) and (17) when $k \to 0$. For a parabolic contour $z = x^2/(2R) - h(t)$ entering liquid at the velocity $U(t) = \dot{h}(t)$, we obtain $a(t) = 2\sqrt{Rh(t)}$, $a\dot{a} = 2RU(t)$ and $\Delta E_j/(\Delta \ell \Delta t) = \frac{1}{2}\pi \rho R U^2(t)$. Taking into account that there are two jets in the two dimensional impact problem, we find the energy flux as

$$\frac{2\Delta E_j}{\Delta \ell \Delta t} = \pi \rho R U^2(t), \tag{22}$$

which coincides with that derived by Molin et al. (1996) in the problem of the parabolic contour entering liquid at constant velocity.

Eq. (21) indicates that in the case of a constant entry velocity and the major semi-axis $b(t) = Bt^{1/3}$, the flux of energy in the jet does not vary in time. Fig. 5 shows the corresponding shape of the body generated with a constant eccentricity $e = 0.9$, $B = 1\,\mathrm{m\,s^{-1/3}}$ and the body velocity ($U = 1\,\mathrm{m\,s^{-1}}$) up to the time instant $t = 0.01$ s. The shape was reconstructed using the technique developed by Scolan and Korobkin (2001). The full shape with disturbed free surface and the shape under the undisturbed free surface are drawn. The length scales are approximately indicated at the tips of the arrows. The free surface elevation χ in Fig. 5 was determined from the kinematic boundary condition $\chi_t(x, y, t) = \phi_z(x, y, 0, t)$, where the derivative $\phi_z(x, y, 0, t)$ outside the contact region is computed using Eq. (1).

Note that the jet cannot be computed within the present analysis. However, we know that the jet is very thin and the jet root is located at the intersection of the liquid free surface and the wet part of the entering body surface. It is seen that the splash ascends the steeper side of the body (boat) more quickly than the bow or stern. For the entry of shape depicted in Fig. 5 into water with specific mass $\rho = 1000\,\mathrm{kg\,m^{-3}}$, the given impact conditions provides approximately constant flux of the energy through the jet. Quantitatively, the flux of the energy is

$$\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t} = 133.2\,\mathrm{kg\ m^2\,s^{-3}} \tag{23}$$

during the initial stage of the process.

Consider the elliptic paraboloid entry problem for which an analytical solution exists. The position of the body at instant t is described by the equation

$$z = \frac{x^2}{A^2} + \frac{y^2}{B^2} - U_0 t, \tag{24}$$

where A^2 and B^2 are length scales in the two horizontal directions. The velocity U_0 of the entering body is constant. In this problem, the contact region is elliptic and expands homothetically, which implies that the flux can be evaluated with

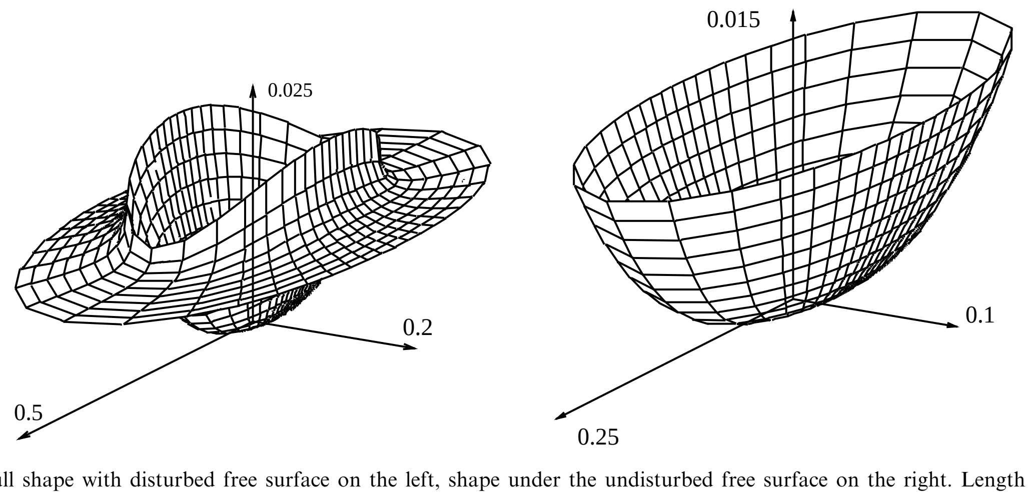

Fig. 5. Full shape with disturbed free surface on the left, shape under the undisturbed free surface on the right. Length scales are indicated at the tip of arrows, the vertical scale is stretched compared to the horizontal ones. The perspective means that indicated length scales do not necessarily fit with the limit of the drawn shape.

the help of Eq. (21). Using the results by Scolan and Korobkin (2001) obtained for an elliptic paraboloid entering liquid at a constant velocity, we find

$$\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t} = \frac{\pi k^2}{2E(e)}\rho U_0^{7/2}B^3\sqrt{t}\left[1 + k^2 D(e)/E(e)\right]^{3/2}, \tag{25}$$

where k is defined by the equation

$$k^2\frac{1 + k^2 D(e)/E(e)}{2 - k^2 D(e)/E(e)} = k_\gamma^2, \tag{26}$$

with $k_\gamma = A/B$ being the aspect ratio parameter which characterizes the slenderness of the entering body. In the axisymmetric case, $A^2 = B^2 = R$, one gets $k_\gamma = 1$, $k = 1$, $e = 0$, $E(0) = \frac{\pi}{2}$, $D(0) = \frac{\pi}{4}$, and Eq. (25) gives

$$\left.\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t}\right|_{k_\gamma=1} = \left(\frac{3}{2}\right)^{3/2}\sqrt{t}\rho U_0^{7/2}R^{3/2}. \tag{27}$$

This formula agrees with that derived by Korobkin (1994) for the axisymmetric case.

Let us consider elliptic paraboloids obtained from the axisymmetric paraboloid, $z = (x^2 + y^2)/R - U_0 t$, by its stretching in horizontal directions without changing the areas of the body cross-sections. This implies that the product AB is the constant equal to R in (24). Therefore,

$$A = \sqrt{k_\gamma R}, \quad B = \sqrt{\frac{R}{k_\gamma}}, \tag{28}$$

where k_γ is now the stretching parameter, $0 < k_\gamma \leq 1$. The two dimensional case corresponds to $k_\gamma = 0$ and the axisymmetric case to $k_\gamma = 1$. It should be noted that the parameter k_γ in Eq. (28) cannot be too small. This follows from the main assumption of the Wagner theory, which requires that the deadrise angle of the entering body be small. Substituting B in Eq. (28) for B in Eq. (25) and taking Eq. (27) into account, we obtain

$$\left.\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t}\right|_{0<k_\gamma\leq 1} = q(k_\gamma)\left.\frac{\mathrm{d}E_j^{\mathrm{tot}}(t)}{\mathrm{d}t}\right|_{k_\gamma=1}, \tag{29}$$

$$q(k_\gamma) = \frac{\pi}{3}\sqrt{\frac{2}{3}}\frac{k^2}{k_\gamma^{3/2}E(e)}\left[1 + k^2 D(e)/E(e)\right]^{3/2}, \tag{30}$$

where the function $k = k(k_\gamma)$ is defined in implicit form by Eq. (26). The function $q(k_\gamma)$ is depicted in Fig. 6. This function characterizes the energy flux for elongated bodies compared to the axisymmetric one of the same volume. It should be noted that $q(k_\gamma) \geq 0.9$ as long as $k_\gamma > 0.43$. This means that the paraboloid must be very elongated—and consequently the Wagner assumption is indeed violated—so that the corresponding flux of energy through the jet significantly deviates from the axisymmetric case. If the body velocity is kept constant during the impact, Eq. (21) can be integrated, which gives the jet energy at instant t as

$$E_j^{\mathrm{tot}}(t) = \frac{1}{2}U_0^2\frac{2\pi\rho a^2 b}{3E(e)}, \tag{31}$$

where the quantity $2\pi\rho a^2 b/(3E(e))$ is known as the added mass $M_a(t)$ of the elliptic disc $D(t)$. Therefore, in the homothetical case and at constant velocity of the body, the jet energy is proportional to the added mass of the expanding disc:

$$E_j^{\mathrm{tot}}(t) = \frac{1}{2}U^2 M_a(t). \tag{32}$$

It is proved in the next section that formula (32) is also valid in nonhomothetical case, when $\mathrm{d}k/\mathrm{d}t \neq 0$.

6. Distribution of kinetic energy

It is well known that the energy conservation law is not satisfied within classical Wagner theory. In the general case,

$$\frac{\mathrm{d}}{\mathrm{d}t}[A(t) - T(t)] = \frac{1}{2}U^2(t)\frac{\mathrm{d}M_a}{\mathrm{d}t}, \tag{33}$$

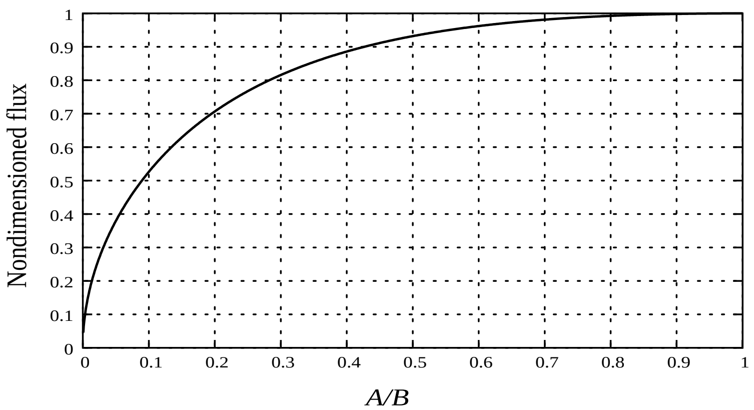

Fig. 6. Variation of the function $q(k_\gamma)$ [see Eq. (30)] with the aspect ratio $k_\gamma = A/B$.

where $T(t)$ is the kinetic energy of the liquid flow in the outer region, $T(t) = \frac{1}{2}M_a(t)U^2(t)$, $A(t)$ is the work done to oppose the hydrodynamic force on the entering blunt body. During the initial stage of the water impact, the added mass of the expanding flat disc $D(t)$ increases, $\mathrm{d}M_a/\mathrm{d}t > 0$. Therefore, Eq. (33) gives $T(t) < A(t)$, which is usually considered as an indication that a part of the energy is ''lost'' during the impact. It is proved below that the flux of energy in the right-hand side of Eq. (33) is equal to the flux of kinetic energy (20) through the jet in the case of elliptic contact lines.

Comparing the right-hand sides of Eqs. (20) and (33), we conclude that it is enough to prove the equality

$$\frac{\mathrm{d}M_a}{\mathrm{d}t} = \frac{2\pi\rho a}{E(e)}\left[\dot{a}b + \frac{1}{3}(\dot{b}a - \dot{a}b)\left(1 + k^2\frac{D(e)}{E(e)}\right)\right], \tag{34}$$

where the added mass $M_a(t)$ of the elliptic disc $D(t)$ is given as $M_a(t) = 2\pi\rho a^2 b/(3E(e))$. By using the following formulae,

$$\frac{\mathrm{d}E}{\mathrm{d}e} = -eD(e), \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \frac{a(a\dot{b} - \dot{a}b)}{b^3 e}, \tag{35}$$

the time derivative of the added mass is found to be

$$\frac{\mathrm{d}M_a}{\mathrm{d}t}(t) = \frac{2\pi\rho}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a^2 b}{E(e)}\right], \tag{36}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a^2 b}{E(e)}\right] = \frac{a}{E(e)}\left[2\dot{a}b + a\dot{b} + ab\dot{e}e\frac{D(e)}{E(e)}\right] = \frac{a}{E(e)}\left[3\dot{a}b + (a\dot{b} - \dot{a}b) + k^2(a\dot{b} - \dot{a}b)\frac{D(e)}{E(e)}\right]. \tag{37}$$

Replacing the latter expression into Eq. (36), we arrive at equality (34). Therefore,

$$\frac{\mathrm{d}E_j^{\mathrm{tot}}}{\mathrm{d}t} = \frac{1}{2}U^2(t)\frac{\mathrm{d}M_a}{\mathrm{d}t} \tag{38}$$

and Eq. (33) provide after its integration with respect to time,

$$A(t) = T(t) + E_j^{\mathrm{tot}}(t). \tag{39}$$

We can conclude now that the energy conservation law is held within the three-dimensional Wagner theory if the jet energy is taken into account. It should be noted that this result is only proved for the case of elliptic contact lines.

It is seen that the energy is equally transmitted to the bulk of the fluid and to the spray jet, $T(t) = E_j^{\mathrm{tot}}(t) = \frac{1}{2}M_a(t)U_0^2$, provided that the velocity of the entering body is constant. If not, one has

$$E_j^{\mathrm{tot}}(t) = T(t) - \int_0^t M_a(\tau)U(\tau)\dot{U}(\tau)\,\mathrm{d}\tau, \tag{40}$$

where $M_a(\tau) \geq 0$ and $U(\tau) > 0$. Therefore, the main part of the energy is transmitted to the bulk of the fluid, $T(t) > E_j^{\mathrm{tot}}(t)$, if the body velocity increases, $\dot{U}(t) > 0$, after the impact instant. Correspondingly, the main part of the energy is transmitted to the jet, $E_j^{\text{tot}}(t) > T(t)$, if the body velocity decreases, $\dot{U}(t) < 0$, after the impact. The velocity of the entering body decreases, in particular, in the case of free fall of the body onto the liquid free surface.

7. Conclusion

The initial stage of vertical impact of a smooth blunt body onto a free surface of ideal and incompressible liquid has been considered. It is shown that the flow description, which is uniformly valid during the initial stage, can be obtained by matching the three-dimensional Wagner solution in the main flow region (outer solution) to the two dimensional nonlinear solution in the jet root region (inner solution). The matching procedure has been justified by a local analysis of the outer solution in the close vicinity of the contact line. It can in fact be proved that the outer solution flow mainly occurs in the normal direction to the contact line in its close vicinity. The analysis is restricted to the case of elliptic contact lines, which are characterized by only two parameters and for which many analytical developments were available.

The uniformly valid solution of the three-dimensional impact problem is used to evaluate the distribution of the energy during the impact process. It is shown that the energy conservation law is satisfied with this combined solution. It was known in both two dimensional and axisymmetric cases that the energy is equally transmitted to the bulk of the fluid and to the spray jet provided that the velocity of the entering body is constant. In the present paper, this result is proved to be valid also for the three-dimensional case.

So far, it is not clear in which way the obtained solution can be modified to account for a nonzero horizontal component of the entering body velocity. One may expect the contact region to be nonelliptic for an elliptic paraboloid entering liquid at an attack angle.

The solutions obtained can also be used to evaluate the uniformly valid pressure distribution over the wet part of the entering body surface. Either the composite solution by Zhao and Faltinsen (1992) or the "second order" solution by Cointe and Armand (1987) can be used. The force is then numerically calculated from pressure integration. This analysis should improve the prediction of the total hydrodynamic force acting on the body. Finally, comparisons with experimental data will determine the field of application of the present formulation, particularly over what time interval the solutions remain valid.

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